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LETTER TO THE EDITOR

Random walks in media with constrained disorder

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Abstract. We have considered in the framework of the field-theoretical approach the asymptotics of random walks in media with quenched disorder which is described by drift velocity field $\boldsymbol{V}(\boldsymbol{x})$ subjected to various constraints. The disorder has proved to be relevant in $d \leqslant 2$ dimensions leading (at $d=2$) to superdiffusion behaviour for the solenoidal random drift ($\partial V_i/\partial x_i = 0$) and to subdiffusion behaviour for the potential random drifts ($\partial V_i/\partial x_j - \partial V_j/\partial x_i = 0$).

A well known problem of random walks in media with a quenched disorder is defined by the Langevin equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{V}(\boldsymbol{x}) + \boldsymbol{\eta}(t). \tag{1}$$

Here $\boldsymbol{\eta}(t)$ is a Gaussian white noise of zero average having the following correlation function

$$\langle \eta_i(t)\eta_j(t')\rangle = 2D_0\delta_{ij}\delta(t-t'), \tag{2}$$

where i, j are the vector indices in d-dimensional space, D_0 is the bare diffusion constant. The equivalent description of the problem (1), (2) is given by the Fokker–Planck equation for the probability distribution $P(\boldsymbol{x}, t)$ of the random walker

$$[\partial/\partial t - \partial_i(D_0\partial_i - V_i)]P(\boldsymbol{x}, t) = 0. \tag{3}$$

In this letter we consider three models for the disorder described by the random drifts $\boldsymbol{V}(\boldsymbol{x})$. For all the models $\boldsymbol{V}(\boldsymbol{x})$ is supposed to be the quenched Gaussian random field of zero average with the correlation function given by

$$\langle V_i(\boldsymbol{x})V_j(\boldsymbol{x}')\rangle = \gamma_0 F_{ij}(\boldsymbol{x}-\boldsymbol{x}') \tag{4}$$

where the Fourier transforms of $F_{ij}(\boldsymbol{x}-\boldsymbol{x}')$ are given as follows:

Model I, isotropic disorder, $\quad F_{ij}(\boldsymbol{k}) = \delta_{ij}$ $\quad$ (5*a*)

Model II, transverse disorder, $\quad F_{ij}(\boldsymbol{k}) = \delta_{ij} - k_ik_j/k^2$ $\quad$ (5*b*)

Model III, longitudinal disorder, $\quad F_{ij}(\boldsymbol{k}) = k_ik_j/k^2.$ $\quad$ (5*c*)

Model I corresponds to the unconstrained random drift field $\boldsymbol{V}(\boldsymbol{x})$. The models II, III correspond to the random vector fields $\boldsymbol{V}(\boldsymbol{x})$ being subject to the following constraints:

Model II, solenoidal field: $\quad \partial_i V_i(\boldsymbol{x}) = 0$ $\quad$ (6*b*)

Model III, potential field: $\quad \partial_i V_j(\boldsymbol{x}) - \partial_j V_i(\boldsymbol{x}) = 0.$ $\quad$ (6*c*)

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In the model III, the random field V may evidently be considred as a gradient of a random potential, $V_i(x) = \partial_i \Phi(x)$.

The recent field-theoretical analysis of the model I (Luck 1983, Peliti 1984, Fisher 1984, Cardy 1984) has shown the weak disorder to be relevant in $d \leqslant 2$ dimensions. At the upper critical dimensionality $d = 2$ the disorder (4) leads to logarithmic corrections to the diffusion coefficient in the long-time limit (Fisher 1984):

$$D(t) \equiv \langle x^2(t) \rangle / t \propto D_0(1 + 4/\ln t). \tag{7}$$

We will show the diffusion coefficient in both models, II and III, to be much more affected (at $d \leqslant 2$) with a weak disorder than in the model I. For $d = 2$ we will obtain in the long-time limit:

$$\text{Model II,} \qquad D(t) \propto (\ln t)^{1/2} \tag{8}$$

$$\text{Model III,} \qquad D(t) \propto t^{-\gamma_0/(8\pi D_0^2)}. \tag{9}$$

Before treating the models II and III let us discuss what physics they describe. The most straightforward realisation of the models (5) is a random motion of a traveller in a liquid with stationary random streams. The travelling in an incompressible liquid ('a bottle in an ocean') corresponds evidently to the model II (6*b*). Travelling in a compressible liquid with random non-circulatory flows corresponds to model III (6*c*).

Another physical problem described in a large-distance limit by the Fokker–Planck equation (3) is the general hopping problem on an irregular lattice:

$$\dot{P}_x = \sum_y (W_{xy} P_y - W_{yx} P_x). \tag{10}$$

Both the continuum (3) and the lattice (10) problems turn out to have the same critical properties because they differ only by irrelevant terms containing the derivatives of higher orders (Derrida and Luck 1983, Luck 1983). Then one can express the parameters of the Fokker–Planck equation (3) in terms of the random hopping rates W_{xy} as follows

$$V(x) = \sum_y (y - x) W_{xy} \tag{11}$$

$$D(x) = \tfrac{1}{2} \sum_y (y - x)^2 W_{xy}. \tag{12}$$

The Gaussian fluctations of the drift field (11) result in the non-trivial critical behaviour in $d \leqslant 2$ dimensions while both non-Gaussian corrections to the correlation function (4) and the random variations of $D(x)$ are irrelevant. In a regular lattice subjected to no external fields the drift velocity (11) equals zero. For the irregularities caused by the presence of charged impurities when changes in hopping rates are proportional to random electric fields the hopping problem turns out to be described by the model III. The model II may arise if the irregularities are caused by the presence of dislocations (vortices).

To derive the announced results (8) and (9) we begin with the Fokker–Planck equation (3). All the physical properties of random walks are governed with the probability density that a walker starting at the origin reaches the point x at time t. This probability density is given by the Green function $G(x, t)$ of the operator on the LHS of the equation (9). In order to perform an averaging of the Green function over the random field $V(x)$ it is convenient to represent the Fourier transform $G(x, \omega)$ as a functional integral over conjugated fields φ, $\bar{\varphi}$ as follows:

$$G(\boldsymbol{x}, \omega) = Z^{-1} \int \mathscr{D}\bar{\varphi}\, \mathscr{D}\varphi\, \varphi(\boldsymbol{x})\bar{\varphi}(0) \exp(\mathrm{i}S) \tag{13}$$

$$S = \int \bar{\varphi}(\boldsymbol{x})[\mathrm{i}\omega - \partial_i(D_0\partial_i - V_i(\boldsymbol{x}))]\varphi(\boldsymbol{x})\, \mathrm{d}^d x \tag{14}$$

with Z being a normalisation factor. The averaging over $\{\boldsymbol{V}(\boldsymbol{x})\}$ is performed with the help of a conventional replica trick and results in substituting the action (14) by the following one

$$S = \int \bar{\varphi}_\alpha(\boldsymbol{x})[\mathrm{i}\omega - D\partial_i^2]\varphi_\alpha(\boldsymbol{x}) + \tfrac{1}{2}\mathrm{i}\gamma \int F_{ij}(\boldsymbol{x}-\boldsymbol{x}')\varphi_\alpha(\boldsymbol{x})\partial_i\bar{\varphi}_\alpha(\boldsymbol{x})\varphi_\beta(\boldsymbol{x}')\partial_j\bar{\varphi}_\beta(\boldsymbol{x}')\, \mathrm{d}^d x\, \mathrm{d}^d x' \tag{15}$$

where the replica (Greek) indices run from 1 to N and N must be put equal to zero in the final results. The equivalence in a perturbative sense of the constructed field theory and the initial problem (9) may be directly verified by a proper expansion in powers of γ (note that for $d = 2$ all the diagrams diverge as powers of $\ln t$). Thus the initial problem is represented in a form allowing a rigorous renormalisation group (RG) treatment.

Usual power counting for the action (15) shows the disorder parameter γ to be relevant in $d \leqslant 2$ dimensions for all the models (5). Following the RG scheme commonly used for analysis of critical phenomena (Ma 1976) we integrate over fast Fourier components of fields φ_α, $\bar{\varphi}_\alpha$ with momenta k, $\lambda k_0 < k < k_0$, where λ is a scaling factor $(0 < \lambda < 1)$, k_0 is an ultraviolet cutoff (which is determined by a lattice spacing for the associated hopping problem or by a required regularisation of the correlator (4) for the continuous problem). The RG equations for both diffusion coefficient D and disorder γ are derived as usual by means of a loop expansion coinciding with an expansion in powers of the dimensionless (for $d = 2$) charge

$$g = \gamma/(4\pi D^2). \tag{16}$$

The considered problem of the weak disorder corresponds to a small value of the parameter: $g_0 \ll 1$.

In $d = 2$ dimensions we obtain up to two-loop order the following RG equations $(N = 0)$:

$$\mathrm{d} \ln D/\mathrm{d}\xi = \alpha g - 2(1-\alpha^2)g^2 \tag{17}$$

$$\mathrm{d} \ln g/\mathrm{d}\xi = -(1+\alpha)g + 2(1-\alpha^2)g^2 \tag{18}$$

where $\xi = \ln \lambda^{-1}$ and

$$\alpha = \begin{cases} 0, & \text{model I} \\ 1, & \text{model II} \\ -1, & \text{model III.} \end{cases} \tag{19}$$

The non-trivial renormalisation of frequency ω is obviously absent as a result of the probability conservation law. On solving the equations (17), (18) we obtain $D(\xi)$ as follows:

$$\begin{array}{ll} \text{model I} & D(\xi)=D_0\exp[-2g_0^2\xi/(1+g_0\xi)] \\ \text{model II} & D(\xi)=D_0(1+2g_0\xi)^{1/2} \\ \text{model III} & D(\xi)=D_0\exp(-g_0\xi). \end{array} \tag{20}$$

Here as usual ξ must be put equal to $\ln(Dk_0^2/\omega)^{1/2} \propto \ln t^{1/2}$. For $g_0 \ln t \gg 1$ we therefore come to the announced results (8), (9) as well as to the expression (7) for the model I.

For $d=2-\varepsilon$ one should add ε to the RHS of the equation (18). It makes the fixed point $g^*=\varepsilon/(1+\alpha)$ to arise in the models I and II thus resulting in universal exponents for diffusion:

$$\begin{array}{ll} \text{model I:} & D \propto t^{-\varepsilon^2} \\ \text{model II:} & D \propto t^{\varepsilon/4}. \end{array}$$

With the accuracy up to two loops one fails to find the fixed point in the model III.

Note that in $d=2$ dimensions the small RG charge decreases under RG transformations ('zero-charge' situation) for both models I and II so that the theory proves to be asymptotically exact. For the model III the renormalisation of the charge g vanishes up to two-loop order. As the straightforward calculations in the three-loop order seem to be unreaslistic the problem of $g(\xi)$ dependence remains undecided for model III.

Now we suggest a simple interpretation of drastic changes in the asymptotics of the random walks affected by the quenched vector disorder with the constraints (6). Figure 1 represents possible realisation of the random drifts $\boldsymbol{V}(\boldsymbol{x})$. The ballistic motion of a walker along the drift lines is perturbed by the strong random 'wind' $\boldsymbol{\eta}(t)$ (1) which blows away the walker from one drift line to another. In model II, the solenoidal field $\boldsymbol{V}(\boldsymbol{x})$ has no sinks and sources (figure 1(b)). Then the clue to the superdiffusion behaviour (see equation (8)) lies in the remnants of the ballistic motion along drift lines. On the contrary, in the model III the random field $\boldsymbol{V}(\boldsymbol{x})$ is potential so that it is characterised by a set of sources and sinks (figure 1(c)). In the absence of the random wind $\boldsymbol{\eta}(t)$ the sinks act as supertraps. The trapping turns out to dominate the

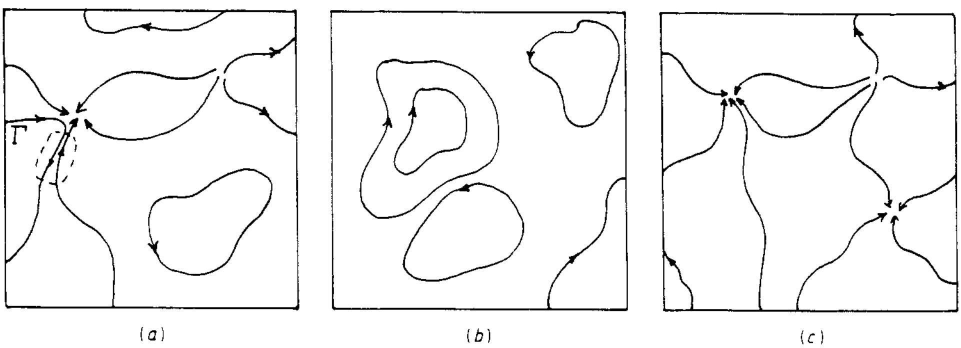

Figure 1. Possible realisations for the random drifts. (a) Unconstrained drifts (model I), (b) solenoidal drifts (model II), (c) potential drifts (model III).

The line Γ (figure 1(a)) which allows a walker to leave a trap does not exist in model III because of non-zero circulation at the dotted contour.

tendency to ballistic motion thus leading to the subdiffusion behaviour (9) even in the presence of the wind. In model I (figure 1(a)), the sinks are not so 'dangerous' for diffusion as in model III due to existence of drift lines such as the line Γ (figure 1(a)) which help the walker leaving a trap. As a result the correction to the diffusion law proves to be small (see equation (7)).

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